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Understanding individual-level change through the basis functions of a latent curve model

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Abstract

Latent curve models have become a popular approach to the analysis of longitudinal data. At the individual level the model expresses an individual's response as a linear combination of what are called 'basis functions' that are common to all members of a population and weights that may vary among individuals. This paper uses differential calculus to define the basis functions of a latent curve model. This provides a meaningful interpretation of the unique and dynamic impact of each basis function on the individual-level response. Examples are provided to illustrate this sensitivity, as well as the sensitivity of the basis functions, to changes in the measure of time.

Understanding individual-level change through the basis functions of a latent curve model

Longitudinal methodology continues to be an active area of quantitative research. Among the methods proposed, latent curve models (Meredith & Tisak, 1990; Willett & Sayer, 1994) are frequently studied and applied. A primary goal in the application of the model is to account for dependencies in the data by assuming that the response is a function of one or more latent characteristics of change that are shared among members of a population but that may vary in magnitude among individuals. More generally, the use of latent characteristics to account for individual-level responses is common and lies at the heart of another major framework for longitudinal data analysis, namely mixed-effects models. Several seminal books and articles concerning the more general approach are available, with variation in technical coverage and applications (Bollen & Curran, 2006; Davidian & Giltinan, 1995; Diggle, Liang, & Zeger, 1994; Goldstein, 2003; Hox, 2002; Kreft & de Leeuw, 1998; Laird & Ware, 1982; Longford, 1993; McArdle, 1988; McCulloch & Searle, 2001; Meredith & Tisak, 1984; 1990; Raudenbush & Bryk, 2002; Singer & Willett, 2003; Skrondal & Rabe-Hesketh, 2004; Snijders & Bosker, 1999; Verbeke & Molenberghs, 2000). Beyond the quantitative literature, several articles have been published in substantive domains that have introduced latent curve models, emphasizing the potential benefits of the technique in a particular area of study (e.g., Duncan & Duncan, 2004; Hedeker, Flay, & Petraitis, 1996; McArdle, 1986; McArdle & Epstein, 1987).

Many commercial software packages have been developed or existing programs extended to carry out estimation of these models, with packages varying in their offerings. These include HLM (Raudenbush, Bryk, Cheong, & Congdon, 2004), LISREL (Jöreskog, & Sörbom, 2006), Mplus (Muthén & Muthén, 1998-2012), Mx (Neale, Boker, Xie, & Maes, 2003), MLwiN

(Rasbash, Steele, Browne, & Prosser, 2004), S-PLUS (Pinheiro & Bates, 2000), SPSS (IBM SPSS Statistics, IBM Corp., 2012), and the SAS procedures PROC MIXED (Littell, Milliken, Stroup, Wolfinger, & Schabenberger, 2006), NLMIXED (Wolfinger, 1999), and GLIMMIX. The models have been considered for both separate and simultaneous analysis of continuous and discrete data (Muthén, 2001; Skrondal & Rabe-Hesketh, 2004), been based on linear and nonlinear functions to describe change or growth in a measured response (Meredith & Tisak, 1984, 1990; Davidian & Giltinan, 1995), and allowed measurement error in the repeated measures (Blozis, 2006; Chan, 1998; Duncan & Duncan, 1996; McArdle, 1988; Sayer & Cumsille, 2001). Individuals may be observed according to the same time points, or at the other extreme, completely unique times of measurement (Blozis, 2004; Blozis & Cudeck, 1998; Jennrich & Schluchter, 1986). If data are not complete, then inference under these models is considered valid if data are missing completely at random or missing at random (Laird, 1988; Little, 1995). Approaches to handling data that are not missing at random have also been developed (Little, 1993; Hedeker & Gibbons, 1998; Xu & Blozis, 2011).

Under a latent curve model, it is assumed that underlying the observed responses for an individual is a latent trajectory, that is, a true curve that is not directly observed (Bollen & Curran, 2006; Meredith & Tisak, 1984, 1990). A latent curve model characterizes the latent response using a weighted combination of what are called 'basis functions' that are typically evaluated according to time. In a given application of the model, a set of weighted basis functions represents a form of change that the underlying response is assumed *a priori* to follow with the chosen form assumed to be shared across all members of a population. Assuming a response follows a quadratic function of time, for instance, the basis functions at a given time point t may be specified by the set $(f_1 = 1, f_2 = t, f_3 = t^2)$, where t represents a measure of time, such as participant's age or number of years following a key event. In a different form of the model the elements of the basis functions are not be specified but rather are estimated from the data in what have been called latent basis curve models (McArdle, 1988; 1998; Meredith & Tisak, 1990). The former model is a type of 'structured' model in that the basis functions are explicitly specified. This is in contrast to a latent basis curve model in which the basis functions are assumed to be unknown and are estimated from the data with some parameter restrictions placed so that the model is identified. In either case, the basis functions are weighted by coefficients that may vary among individuals to allow for differences in the individuals' dependencies on the basis functions. For a quadratic function, for example, the basis functions $f_1 = 1, f_2 = t$, and $f_3 = t^2$ may be weighted by a set of individual-specific coefficients β_{0i}, β_{1i}, and β_{2i}, respectively. At the population level the response is assumed to follow the common function with fixed coefficients, such as β_0, β_1, and β_2 in the case of the quadratic function.

Although applications of latent curve models often involve fitting the data by using a polynomial function, nonlinear functions not based on a polynomial function have been applied as well (e.g., Choi, Harring, & Hancock, 2009). In the latent curve model developed in Meredith and Tisak (1984; 1990), a nonlinear function is used in which the nonlinear function coefficients are fixed effects. Only coefficients that enter linearly may be random. An exponential function involving a basis function specified as $\exp\{\gamma t\}$, for example, includes a fixed nonlinear coefficient γ that serves as a common weight applied to the measure of time t. This basis function may then be weighted by an individual-specific coefficient β_{0i} that enters the function linearly, where β_{0i} represents the expected response for an individual at $t = 0$: $\beta_{0i}\exp\{\gamma t\}$. Note that this is in contrast to a nonlinear mixed-effects model in which random coefficients may enter the function in a linear or nonlinear manner (Davidian & Giltinan, 1995).

Related to the latent curve model is what Browne and Du Toit (1991; also see Browne, 1993) called a structured latent curve model. In a structured latent curve model, the mean response is assumed to follow a specific function that may include parameters that enter nonlinearly and all of which are fixed. The function is referred to as the ‘target’ function and is common to all individuals. Individual-level responses are then assumed to follow a function that is defined by a first-order Taylor expansion taken with respect to the coefficients of the target function and evaluated according to time. An important aspect of the structured latent curve model is that the population model is invariant to a constant scaling factor. Under this condition, the mean structure may be decomposed into 1) a matrix formed by the first-order partial derivatives of a common function taken with regard to its coefficients and 2) a set of fixed weights (cf: Shapiro & Browne, 1987, Condition 2). The weights that combine with the basis functions enter the model linearly. Any coefficient entering the model in a nonlinear manner, such as those that define one or more of the basis functions, must be fixed.

The purpose of this paper is to show how differential calculus can be used to extend one’s understanding of a latent curve model beyond the usual interpretation that focuses on the mean response and individual differences in the growth coefficients. That is, most applications of a latent curve model report on the average response and may have some discussion around individual differences in the characteristics that describe change (i.e., the growth coefficients). Our focus here is on the individual curves themselves and showing how one can use information from the model to better understand growth or change at the individual level. As we illustrate, a latent curve model may be formulated in a manner similar to a structured latent curve model. That is, the common function of a latent curve model is naturally invariant to a constant scaling factor, and so the individual-level model of a latent curve model may also be defined by a first-order Taylor expansion based on the target function that defines the mean response and first-order partial derivatives taken with respect to the coefficients of the target function. A benefit of defining a latent curve model from this perspective is that it formalizes the meaning of the basis functions of the model as each derivative has a specific interpretation whose impact on the individual's response may change with time.

The remainder of the paper is organized as follows: First, the fundamental features of the latent curve model and structured latent curve model are reviewed. Readers familiar with the technical aspects of these models may bypass this material. Based on the ideas underlying a structured latent curve model, a latent curve model is then defined using differential calculus. That is, a latent curve model is defined by a first-order Taylor expansion of a common function with respect to the coefficients of the growth function. Using longitudinal data from an empirical investigation, the individual-level fitted responses are studied with regard to their dependence on the basis functions that define the model. This approach provides a deeper understanding of a latent curve model in the context of an empirical example.

Latent Curve Model

Let a set of responses for individual *i* be denoted by $\mathbf{y}_i = (y_1, \ldots, y_{n_i})'$ which is to be evaluated according to a given set of time points $\mathbf{t}_i = (t_1, \ldots, t_{n_i})'$, where n_i is the number of measures for the individual. The length of and elements that make up $\mathbf{t}_i$ may vary among individuals. For the individual, a latent curve model assumes the responses are due to a linear combination of a factor matrix $\mathbf{\Lambda}_i$ and a weight vector $\boldsymbol{\eta}_i$, plus a set of residuals $\boldsymbol{\varepsilon}_i$:

$$\mathbf{y}_i = \mathbf{\Lambda}_i \boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i, \quad (1)$$

where $\mathbf{\Lambda}_i$ is an $n_i \times K$ matrix of K basis functions, $\boldsymbol{\eta}_i = (\eta_{1i}, \ldots, \eta_{Ki})'$ is a set of K weights that vary across individuals, and $\boldsymbol{\varepsilon}_i = (\varepsilon_1, \ldots, \varepsilon_{n_i})'$ is an $n_i \times 1$ set of time-specific residuals. The matrix $\mathbf{\Lambda}_i$ may be a function of time and possibly a set of fixed coefficients $\boldsymbol{\gamma} = (\gamma_1, \ldots)'$: $\mathbf{\Lambda}_i(\mathbf{t}_i, \boldsymbol{\gamma})$, where $\boldsymbol{\gamma}$, if included in a model, contributes to the characterization of the mean trajectory. The elements of $\boldsymbol{\gamma}$ may enter $\mathbf{\Lambda}_i$ linearly or nonlinearly but are fixed across individuals. Although the particular functions that make up the columns of $\mathbf{\Lambda}_i$ are assumed to be common across individuals, the matrix may vary among individuals with regard to time $\mathbf{t}_i$, hence the use of the subscript i **for** Λ_i. The set of weights in $\boldsymbol{\eta}_i$ may vary among individuals to allow individual differences in the dependencies on the basis functions contained in $\mathbf{\Lambda}_i$.

The time-specific residuals are often assumed to be i.i.d. normal with mean equal to $\mathbf{0}$ and symmetric covariance matrix $\mathbf{\Theta}_i(\boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\theta_1, \ldots)'$ contains the unique parameters of the matrix. The matrix $\mathbf{\Theta}_i$ is $n_i \times n_i$ and so can vary among individuals in terms of its dimensions, so as to allow for individuals to be measured a different number of times, but often not otherwise. Usually $\mathbf{\Theta}_i$ is parameterized to represent a specific pattern in the residuals, such as residuals that are independent between occasions with constant variance across time or that are autocorrelated across time. Although it is often necessary to impose a structure on Θ_i for model identification purposes, specific patterns for $\mathbf{\Theta}_i$ may be considered to test specific hypotheses about the residuals. Among individuals, the set of random weights $\boldsymbol{\eta}_i$ is assumed to be i.i.d. normal with mean $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_K)'$ and symmetric covariance matrix $\mathbf{\Phi}(\boldsymbol{\phi})$, where $\boldsymbol{\phi} = (\phi_1, \ldots)'$ contains the unique parameters of the matrix. In most applications the matrix $\mathbf{\Phi}$ is assumed to be unstructured so as to not impose restrictions on the variances of the random weights and to allow covariances between them.

Given the distributional assumptions for $\boldsymbol{\varepsilon}_i$ and $\boldsymbol{\eta}_i$ and assuming independence between them, the marginal mean and covariance structure of a latent curve model are

$$\boldsymbol{\mu}_i = \boldsymbol{\Lambda}_i(\mathbf{t}_i, \boldsymbol{\gamma})\boldsymbol{\alpha}$$

and

$$\boldsymbol{\Sigma}_i = \boldsymbol{\Lambda}_i(\mathbf{t}_i, \boldsymbol{\gamma})\boldsymbol{\Phi}\boldsymbol{\Lambda}_i(\mathbf{t}_i, \boldsymbol{\gamma})' + \boldsymbol{\Theta}_i,$$

respectively. The dimensions of $\boldsymbol{\mu}_i$ and $\boldsymbol{\Sigma}_i$ are n_i. Both $\boldsymbol{\Phi}$ and $\boldsymbol{\Theta}_i$ are taken to be positive definite. The parameters of the model are $\boldsymbol{\gamma}$, $\boldsymbol{\alpha}$, $\boldsymbol{\phi}$, and $\boldsymbol{\theta}$, where some or all of the elements in $\boldsymbol{\gamma}$ and $\boldsymbol{\alpha}$ may be shared between them (see the example provided later). Maximum likelihood (ML) estimation may be carried out to obtain parameter estimates and their standard errors, as well as confidence intervals for the parameters using methods that are traditionally used to fit linear models (e.g., Jennrich & Schluchter, 1986).

Structured Latent Curve Model

Now, let a set of responses for individual *i* be denoted by $\mathbf{y}_i = (y_1, \ldots, y_n)'$ where it is assumed for the sake of the discussion here that the times of measurement $\mathbf{t} = (t_1, \ldots, t_n)'$ are identical for all individuals. This is not a requirement of the structured latent curve model but serves to simplify the discussion here without loss of generality. Browne (1993) and Browne and Du Toit (1991) developed the model assuming that the times of measurement were identical for all individuals, and Blozis (2004) extended the model to allow for the times of measurement to differ among individuals.

In a structured latent curve model, the set of expected values of the response $\mathbf{y}_i$,

$\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)'$, observed according to **t** is assumed to follow a smooth function of **t** and a set of fixed coefficients $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_K)'$:

$$\boldsymbol{\mu} = \mathbf{f}(\mathbf{t}, \boldsymbol{\gamma}) \tag{2}$$

where $\mathbf{f}(\cdot)$ is the target function and K denotes the number of fixed coefficients. In Browne (1993), only monotonic target functions were considered. The function in (2) is assumed to be smooth such that for each evaluation of the target function at the values of **t** the function is differentiable with respect to parameter γ at its population value γ_o.

At the individual level the observed response is assumed to be the sum of a true score and error, where the true score is defined by a linear weighted combination of a set of basis functions. Specifically, the set of observations for an individual is defined by using a first-order Taylor expansion based on the chosen target function **defined for the mean response**:

$$\mathbf{y}_i = \mathbf{f}(\mathbf{t}, \boldsymbol{\gamma}) + \boldsymbol{\Lambda}(\mathbf{t}, \boldsymbol{\gamma})\boldsymbol{\eta}_i^* + \boldsymbol{\varepsilon}_i, \tag{3}$$

where

$$\boldsymbol{\Lambda}(\mathbf{t}, \boldsymbol{\gamma}) = \begin{bmatrix} \frac{\partial f(t_1, \boldsymbol{\gamma})}{\partial \gamma_1} & \cdots & \frac{\partial f(t_1, \boldsymbol{\gamma})}{\partial \gamma_K} \\ \vdots & \ddots & \vdots \\ \frac{\partial f(t_n, \boldsymbol{\gamma})}{\partial \gamma_1} & \cdots & \frac{\partial f(t_n, \boldsymbol{\gamma})}{\partial \gamma_K} \end{bmatrix}, \tag{4}$$

with $\frac{\partial f(t_j, \boldsymbol{\gamma})}{\partial \gamma_k}$ being the first-order partial derivative of the target function with regard to the k^{th} coefficient in $\boldsymbol{\gamma}$ and evaluated at time t_j, where $j = 1, \ldots, n$. Although the target function $\mathbf{f}(\mathbf{t}, \boldsymbol{\gamma})$ in (2) is chosen to be monotonic (see the Appendix for details describing a structured latent curve model for a non-monotonic function), the functions that make up the columns of the matrix $\mathbf{\Lambda}$ need not be monotonic (see Browne, 1993). The model for the individual-level response in (3) also need not be monotonic even if the target function is assumed to be monotonic. The coefficient vector $\boldsymbol{\eta}_i^* = (\eta_{1i}^*, \ldots, \eta_{Ki}^*)'$ is a set of random, individual-level weights that are assumed to have an expected value of 0, satisfying an assumption of the structured latent curve model that the expected value of $\mathbf{y}_i$ is equal to $\mathbf{f}(\mathbf{t}, \boldsymbol{\gamma})$. Thus, although the average response is assumed to follow the specified target function, the responses at the individual level can in fact be quite different, as will be illustrated in the example that follows. As a note, $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}_i^*$ are shown here to have the same number of elements, although it is not necessary that each fixed effect has a random effect.

If the model for the mean response is invariant to a constant scaling factor (Shapiro & Browne, 1987, Condition 2; also see Browne, 1993), the mean function can be decomposed into a linear combination of the factor matrix $\mathbf{\Lambda}(\mathbf{t}, \boldsymbol{\gamma})$ and a set of fixed coefficients $\boldsymbol{\alpha}$ (cf: Browne, 1993):

$$\mathbf{f}(\mathbf{t}, \boldsymbol{\gamma}) = \mathbf{\Lambda}(\mathbf{t}, \boldsymbol{\gamma})\boldsymbol{\alpha}, \tag{5}$$

where $\boldsymbol{\alpha}$ may be obtained by solving the set of linear equations in (5) (Shapiro & Browne, 1987). This is an important note about the choice of a target function for a structured latent curve model. That is, if the target function can be decomposed into this linear combination in (5), then the structured latent curve model is a latent curve model in a strict sense, as defined by Meredith and Tisak (1990) (Browne, 1993, p. 7). As shown in Shapiro and Browne, a mean function that can be decomposed into a linear combination as in (5) can be shown to have a target function that is invariant to a scaling factor, that is: $\mathbf{f}(\mathbf{t}, \boldsymbol{\gamma}^*) = \kappa\mathbf{f}(\mathbf{t}, \boldsymbol{\gamma})$ where $\boldsymbol{\gamma}^* = \kappa\boldsymbol{\gamma}$.

Browne (1993) describes several functions that meet this condition, including the Richards function that subsumes the monomolecular, logistic and Gompertz functions (Richards, 1959). Given a target function that meets this criteria, the model will meet the requirement of a latent curve model that the true response is a linear weighted combination of its corresponding basis functions. It is important to note that the first-order Taylor expansion that defines a structured latent curve model represents a direct decomposition of the common function, and as such, does not provide a linear approximation to the function.

Given a target function for the mean response that can be decomposed into the linear combination, the model for the individual response in (3) can be re-expressed by substitution of the mean function $\mathbf{f}(\mathbf{t},\boldsymbol{\gamma})$ in (3) by the expression in (5):

$$\mathbf{y}_i = \mathbf{f}(\mathbf{t},\boldsymbol{\gamma}) + \boldsymbol{\Lambda}(\mathbf{t},\boldsymbol{\gamma})\boldsymbol{\eta}_i^* + \boldsymbol{\varepsilon}_i, \quad \text{(3) repeated}$$

$$\mathbf{y}_i = \boldsymbol{\Lambda}(\mathbf{t},\boldsymbol{\gamma})\boldsymbol{\alpha} + \boldsymbol{\Lambda}(\mathbf{t},\boldsymbol{\gamma})\boldsymbol{\eta}_i^* + \boldsymbol{\varepsilon}_i. \quad (6)$$

Then, assuming $\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\eta}_i^*$, the model for $\mathbf{y}_i$ in (6) may be simplified to

$$\mathbf{y}_i = \boldsymbol{\Lambda}(\mathbf{t},\boldsymbol{\gamma})\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i,$$

The set of individual weights $\boldsymbol{\eta}_i$ allows individuals to vary in their dependencies on the basis functions. As a result, an individual's fitted curve need not be of the same form as that specified for the mean curve. This is a notable feature of a structured latent curve model that distinguishes it from a nonlinear mixed-effects model. For a nonlinear mixed-effects models in which one or more of the random effects enter the model in a nonlinear way, the mean of the individual trajectories is not necessarily equal to the mean trajectory, that is, $E[\boldsymbol{\Lambda}_i(\mathbf{t}_i,\boldsymbol{\gamma})\boldsymbol{\eta}_i] \neq f(\mathbf{t}_i,\boldsymbol{\gamma})$ (see e.g., Cudeck & Harring, 2007). Conversely, for a structured latent curve model, the random effects enter the model in a linear way, and a consequence of this is that the mean of the individual trajectories are actually equal to the mean trajectory.

Assumptions about the individual-level weights and the time-specific residuals are analogous to those discussed earlier for the latent curve model. Given these distributional assumptions for $\boldsymbol{\eta}_i$ and $\boldsymbol{\varepsilon}_i$ and assuming independence between them, the marginal mean and covariance structure of a structured latent curve model, now allowing for unique data collection schemes at the individual level, are $\boldsymbol{\mu}_i = \boldsymbol{\Lambda}_i(\mathbf{t}_i, \boldsymbol{\gamma})\alpha$ and $\boldsymbol{\Sigma}_i = \boldsymbol{\Lambda}_i(\mathbf{t}_i, \boldsymbol{\gamma})\boldsymbol{\Phi}\boldsymbol{\Lambda}_i(\mathbf{t}_i, \boldsymbol{\gamma})' + \boldsymbol{\Theta}_i,$ respectively, where $\boldsymbol{\Lambda}_i$ may vary among individuals due to the n_i specific times of measurement for individual *i*, and similarly, $\boldsymbol{\Theta}_i$ may vary with regard to its dimensions according to n_i (Blozis, 2004). The dimensions of $\boldsymbol{\mu}_i$ and $\boldsymbol{\Sigma}_i$ are n_i. Both $\boldsymbol{\Phi}$ and $\boldsymbol{\Theta}_i$ are taken to be positive definite. The parameters of the model are $\boldsymbol{\gamma}$, $\boldsymbol{\alpha}$, $\boldsymbol{\phi}$, and $\boldsymbol{\theta}$, where some or all of the elements in γ and α may be shared. Estimation of the model may be carried out using methods that are also used to estimate a latent curve model.

Using Differential Calculus to Define the Basis Functions of a Latent Curve Model

The first-order Taylor expansion that defines the individual response under a structured latent curve model can be used more generally to define a latent curve model. That is, a latent curve model is based on a decomposition of a matrix of basis functions and a set of random coefficients. Without a loss in interpretation, it is assumed that all individuals are observed according to the same values of time $\mathbf{t} = (t_1, \ldots, t_n)'$, where *n* is a common number of observations for all individuals. An individual's response $\mathbf{y}_i$ is assumed to be the sum of a true score $\mathbf{c}_i$ and error $\boldsymbol{\varepsilon}_i$:

$$\mathbf{y}_i = \mathbf{c}_i + \boldsymbol{\varepsilon}_i.$$

At the population level the response is assumed to depend on a common function given by

$$\boldsymbol{\mu} = \mathbf{f}(\mathbf{t}, \boldsymbol{\gamma}), \tag{7}$$

where $\boldsymbol{\mu}$ is the expected value of the set of responses across individuals and time $\mathbf{t}$. A set of fixed coefficients $\boldsymbol{\gamma}$ includes elements that may enter the function in a linear or nonlinear way. Following Browne (1993), the mean response in a latent curve model is assumed to follow a smooth curve such that $\mathbf{f}(\mathbf{t}, \boldsymbol{\gamma})$ has continuous derivatives with respect to $\boldsymbol{\gamma}$ that are non-zero for all values in $\mathbf{t}$. The true score at the individual level may then be a re-expression of the common function $\mathbf{f}(\mathbf{t}, \boldsymbol{\gamma})$ as a first-order Taylor expansion:

$$\mathbf{y}_i = \mathbf{f}(\mathbf{t}, \boldsymbol{\gamma}) + \mathbf{f}_1^{'} \eta_{1i}^{*} + \ldots + \mathbf{f}_K^{'} \eta_{Ki}^{*} + \boldsymbol{\varepsilon}_i, \tag{8}$$

where $\mathbf{f}_k^{'}$ is the first-order partial derivative of the common function with regard to the kth coefficient in $\boldsymbol{\gamma}$, evaluated according to $\mathbf{t}$:

$$\mathbf{f}_k^{'} = \frac{\partial \mathbf{f}(\mathbf{t}, \boldsymbol{\gamma})}{\partial \boldsymbol{\gamma}_k},$$

for $k = 1, \ldots, K$. As given earlier, the set of random coefficients in $\boldsymbol{\eta}_i^{*} = (\eta_{1i}^{*}, \ldots, \eta_{Ki}^{*})'$ are the person-level weights with expected values equal to 0. A factor matrix is then defined by the set of K basis functions evaluated according to $\mathbf{t}$, as in (4).

Within the domain of the common function $\mathbf{f}(\mathbf{t}, \boldsymbol{\gamma})$, the derivative of the function with respect to γ_m evaluated at t_j is

$$\frac{\partial f(t_j,\boldsymbol{\gamma})}{\partial \gamma_m} = \lim_{h \to 0}[(f(t_j,\gamma_1,\ldots,\gamma_m+h,\ldots,\gamma_M) - f(t_j,\gamma_1,\ldots,\gamma_m,\ldots,\gamma_M))/h],$$

where h is a small number with a limiting value of 0. Thus, the value of $\frac{\partial f(t_j,\boldsymbol{\gamma})}{\partial \gamma_m}$ is the derivative at γ_m and t_j of the function of one coefficient obtained by holding $\gamma_1,\ldots,\gamma_{m-1},\gamma_{m+1},\ldots,\gamma_M$ fixed and by considering $f(t_j,\boldsymbol{\gamma})$ to be a function of γ_m only. In other words, the partial derivative of the function with regard to γ_m is interpreted as the rate of change in the mean response, as defined by the response function, with regard to γ_m at t_j, holding constant all other coefficients in γ.

To illustrate this, we consider a latent curve model that is defined by a quadratic polynomial function. Assuming a quadratic growth model for the population, a common function may be specified as

$$f(t_{ij},\boldsymbol{\gamma}) = \gamma_0 + \gamma_1 t_{ij} + \gamma_2 t_{ij}^2 \qquad \text{for } j = 1,\ldots,n_i,$$

where n_i is the number of observations for individual i. A set of basis functions is defined by the first-order partial derivatives of the function with regard to the intercept γ_0, instantaneous change rate γ_1, and acceleration rate γ_2:

$$f_1' = \partial f(t_{ij},\boldsymbol{\gamma})/\partial \gamma_0 = 1, \tag{9a}$$

$$f_2' = \partial f(t_{ij},\boldsymbol{\gamma})/\partial \gamma_1 = t_{ij}, \tag{9b}$$

and

$$f_3' = \partial f(t_{ij},\boldsymbol{\gamma})/\partial \gamma_2 = t_{ij}^2, \tag{9c}$$

respectively. The corresponding matrix of basis functions defined by (9a), (9b), and (9c) and evaluated according to t_i is

$$\mathbf{\Lambda}(\mathbf{t}_i) = \begin{bmatrix} 1 & t_1 & t_1^2 \\ \vdots & \vdots & \vdots \\ 1 & t_{n_i} & t_{n_i}^2 \end{bmatrix}.$$

Let $\mathbf{t} = (0, 1, 2, 3)'$ for an individual with complete data at 4 occasions, for instance. Each basis function makes a unique contribution to the characterization of the individual response trajectory. The contribution of the intercept basis function, $\mathbf{f}_1'$, is constant across time. Thus, this function exerts a constant influence over the response. That of the linear slope basis function, $\mathbf{f}_2'$, changes at a constant rate; with passing time, this function has a steady and increasing influence on the trajectory. Note that the basis function of the linear slope in a quadratic growth model is dependent on the point of origin for t and will change given a different point of origin. Finally, the impact of the quadratic slope basis function, $\mathbf{f}_3'$, varies over time; that is, this function has an accelerating, increasing impact on the projected trajectory with the progression of time. As partial derivatives, each basis function represents the sensitivity of the response with regard to the specific features that describe the response over time.

The basis functions in turn may be sensitive to time. This may be assessed by the partial derivatives of the basis functions that define $\mathbf{\Lambda}(\mathbf{t}_i)$ with regard to $\mathbf{t}_i$. Considering the basis functions in (9a), (9b), and (9c), the partial derivatives of the basis functions with respect to $\mathbf{t}_i$ are

$$\mathbf{g}_1 = \partial \mathbf{f}_1' / \partial \mathbf{t}_i = \mathbf{0},$$

$$\mathbf{g}_2 = \partial \mathbf{f}_2^{'} / \partial \mathbf{t}_i = \mathbf{1},$$

and

$$\mathbf{g}_3 = \partial \mathbf{f}_3^{'} / \partial \mathbf{t}_i = 2\mathbf{t}_i,$$

respectively. For the quadratic growth model, the first basis function is not sensitive to the measure of time. The sensitivity of the second is constant across time. That of the third increases at a constant rate at two times the value of $\mathbf{t}_i$, indicating a steady increase in the sensitivity of the function with regard to time.

Empirical Example

To illustrate how change in longitudinal measures under a latent curve model may be better understood through a study of the basis functions of the model, data are presented from a computerized learning task that simulated the activities of an air traffic controller (Kanfer & Ackerman, 1989). The task was continuous, and the response represented the number of planes brought in to land safely every ten minutes. Subjects were allowed 10-minute breaks following the completion of each of three subsequent trials after the initial trial (i.e., after Trial 4 and Trial 7) to minimize practice effects. The researchers administered this task to multiple samples. The sample studied here consists of N = 140 participants who carried out the task continuously for a period of 100 minutes, yielding 10 scores. Scores from the first trial were discarded allowing for an adjustment period to the task. Observed scores by trial for an arbitrary subset of 20 individuals are displayed in Figure 1. These data have appeared in a number of publications that have focused on repeated measures data marked by individual differences in multiple aspects of the longitudinal profile (e.g., Browne,1993; Choi, Harring, &

Hancock, 2009). As will become evident, this particular example data set from a learning study is useful in understanding the role of the basis functions of a latent curve model on the individual-level responses.

Browne (1993) presented three target functions: (i) an exponential function, (ii) a logistic function, and (iii) a Gompertz function, for a structured latent curve model, each of which was thought to be suitable in describing a longitudinal response in which the average response followed a negatively accelerated, monotonic curve. The three functions were parameterized so that the parameters of each of the functions shared the same interpretation. Here, responses to the flight simulation task were fit using the exponential and logistic functions. The exponential function is described next. Readers may refer to Browne for details concerning the logistic function.

For the data from the computerized learning task, the mean trial scores were assumed to follow a smooth, negatively accelerated exponential function (cf: Meredith & Tisak, 1990), where for trial t, the model for the mean response μ_t was

$$\mu_t = f(trial, \boldsymbol{\gamma}) = \gamma_1 - (\gamma_1 - \gamma_2)\exp(-\gamma_3(trial_t - 1)). \tag{10}$$

Elements of the coefficient set $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)'$ denote the mean potential performance score at asymptote, the mean score at the first trial, and the mean rate of change in performance, respectively. At the individual level, the responses were assumed to follow a first-order Taylor expansion of the target function in (10):

$$y_{ij} = f(trial_t, \boldsymbol{\gamma}) + \eta^*_{1i}f'_1 + \eta^*_{2i}f'_2 + \eta^*_{3i}f'_3 + \varepsilon_{ij}, \tag{11}$$

where

$$f_1' = 1 - \exp\{-\gamma_3(trial_t - 1)\},$$

$$f_2' = \exp\{-\gamma_3(trial_t - 1)\},$$

and

$$f_3' = (\gamma_1 - \gamma_2) \cdot (trial_t - 1) \cdot \exp\{-\gamma_3(trial_t - 1)\}$$

are the first-order partial derivatives of the target function taken with regard to γ_1, γ_2, and γ_3, respectively, and evaluated at $trial_t - 1$. In (11) the coefficients $\eta^*_{1i},\ \eta^*_{2i},$ and η^*_{3i} are person-level weights that are applied to each basis function and represent a particular change feature. The trial-specific residual is given by ε_{ij}.

Table 1 provides information on model fit for the structured latent curve model using the exponential and the logistic function and applying each using four different level-1 error covariance structures. Based on the Bayesian information criterion (BIC), performance scores may be best characterized by the exponential function with the AR(1) error covariance structure, although model fit for the logistic function also using the AR(1) error structure is not appreciably different. For the results presented here, the model based on the exponential function was provisionally accepted. Individual log-likelihood function values are plotted in Figure 2 under these two models to examine the relative fit of the model at the person level. Large values of the individual log-likelihood function indicate cases that were not fit as well relative to cases with small function values. As can be seen in Figure 2, there was little discrepancy in model fit between the two functions for the majority of cases. For the few cases that had the most extreme log-likelihood function values relative to other cases there was less agreement between the two functions.

ML estimates with 95% confidence intervals for the fixed effects and estimated variances and covariances for the sample data are provided in Table 2. All 95% confidence intervals for the fixed effects did not include 0. Examination of the fixed effects shows that the average performance level was 20.18 at the initial trial and showed considerable improvement, reaching a plateau of 40.64 at later trials. This suggests the potential for the average number of planes to be landed safely is approximately double the value at the start of the learning trials. The rate of improvement in the mean task performance was 0.31.

Next, individual differences in the three aspects of performance can be ascertained by examining the estimated variances and covariances of the person-level weights:

$$\widehat{\mathbf{\Phi}} = \begin{pmatrix} 78.28 & & \\ 18.91 & 80.50 & \\ -0.70 & 1.74 & 0.58 \end{pmatrix}.$$

The associated correlation matrix of the person-level weights is

$$\mathbf{D}_{\phi}^{-1/2}\widehat{\mathbf{\Phi}}\mathbf{D}_{\phi}^{-1/2} = \begin{pmatrix} 1 & & \\ 0.238 & 1 & \\ -0.261 & 0.638 & 1 \end{pmatrix}$$

where $\mathbf{D}_{\phi} = diag(\widehat{\mathbf{\Phi}})$. These values suggest that those individuals who began the trials with higher performance scores ended the trials with higher performance scores, although this relation is fairly weak ($\hat{\rho}_{21} = 0.238$). Not surprisingly perhaps was the fact that individuals who began with lower initial performance scores tended to gain faster over subsequent trials ($\hat{\rho}_{31} = -0.261$).

Of particular interest are the person-level weights and how these values impact the fit of a model to an individual's responses. To study this further, ML estimates of the model parameters, $\boldsymbol{\alpha}$, $\boldsymbol{\phi}$, and $\boldsymbol{\theta}$, were used to predict the person-level weights, η_i^* for the sample. Once empirical Bayes estimates of the person-level weights, $E[\boldsymbol{\eta}_i^*|\mathbf{y}_i]$, have been computed, an individual's coefficients are simply the sum of the fixed effects in $\boldsymbol{\alpha}$ and the predicted person-level weights:

$$\hat{\boldsymbol{\eta}}_i = \hat{\boldsymbol{\alpha}} + \hat{\boldsymbol{\eta}}_i^*.$$

Computation of the person-level weights for individual i requires the joint distribution of $\mathbf{y}_i$ and $\boldsymbol{\eta}_i^*$

$$\begin{aligned} \mathbf{f}(\mathbf{y}_i, \boldsymbol{\eta}_i^*) &= \mathbf{f}(\mathbf{y}_i|\boldsymbol{\eta}_i^*)f(\boldsymbol{\eta}_i^*) \\ &= \mathbf{f}(\boldsymbol{\eta}_i^*|\mathbf{y}_i)f(\mathbf{y}_i) \end{aligned}$$

Thus, using Bayes' theorem, for individual i the expected value of $\boldsymbol{\eta}_i^*$ given $\mathbf{y}_i$ is defined as (cf: Fitzmaurice, Laird, & Ware, 2011)

$$\begin{aligned} \hat{\boldsymbol{\eta}}_i^* &= E[\boldsymbol{\eta}_i^*|\mathbf{y}_i] = f^{-1}(\mathbf{y}_i)\int_{\boldsymbol{\eta}_i^*} \boldsymbol{\eta}_i^* f(\mathbf{y}_i|\boldsymbol{\eta}_i^*)f(\boldsymbol{\eta}_i^*)d\boldsymbol{\eta}_i^* \\ &= \widehat{\boldsymbol{\Phi}}\widehat{\boldsymbol{\Lambda}}_i'\widehat{\boldsymbol{\Sigma}}^{-1}(\mathbf{y}_i - \widehat{\boldsymbol{\Lambda}}_i\hat{\boldsymbol{\alpha}}). \end{aligned}$$

As an example, the regression coefficients based on the predicted person-level weights for individual 44 are $\hat{\boldsymbol{\eta}}_{44} = (16.794, 35.303, -.341)'$. Figure 3 shows the data and fitted function for individual 44 superimposed on a graph with the fitted mean function on the empirical means. Figure 3 emphasizes the fact that the curve of an individual need not have the same form as the target function and is still fit well by the model. That is, individual 44 has a fitted curve that is not monotonic, whereas the curve for the mean response is assumed to be monotonic. We see from this that the person-specific weights, as applied to the set of common basis functions, help to dictate the shape of the individual's fitted curve.

To understand further how an individual's fitted function is constructed and interpreted, it is instructive to compute the fitted basis functions, which are shown in Table 3. The column of Table 3 with heading $\widehat{\mathbf{f}}_1'$ represents the initial performance basis function that decreases monotonically to a lower asymptote. The column with heading $\widehat{\mathbf{f}}_2'$ represents the potential performance (asymptote) basis function that starts at zero and increases monotonically to an upper asymptote. The rate of change basis function (e.g., column with heading $\widehat{\mathbf{f}}_3'$) begins at zero, increases rapidly during earlier trials, and then decreases at a slower rate towards a lower asymptote of zero. These function values taken across time highlight the influence that each has across trials. The basis function relating to initial performance has its greatest impact at the start of the trials, and then the effect attenuates with the passing of trials. The basis function relating to the potential performance has no impact on the initial trial and begins its influence by the second trial, with its effect slowly increasing with the passing of trials and reaching its greatest function value at the last trial. The basis function relating to the change rate also exerts not influence at the initial trial and its impact is greatest during the earlier trials relative to later trials. Thus, we can see that each basis function plays a unique role in dictating the course for an individual.

Now using the calculated basis functions (again, common to all individuals), the fitted scores for each person can be calculated by combining the basis function values and the person-specific weights. At the initial trial, the fitted function value of individual 44 depicted in Figure 3 is calculated as

$$\hat{y}_{44,1} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 16.794 \\ 35.303 \\ -0.341 \end{pmatrix} = 16.794.$$

From this we see that the asymptote and rate of change basis functions do not impact the initial value. In contrast, at the 5^{th} trial, the fitted value of individual 44 is computed as

$$\hat{y}_{44,5} = \begin{pmatrix} 0.29 & 0.71 & 23.59 \end{pmatrix} \begin{pmatrix} 16.794 \\ 35.303 \\ -0.341 \end{pmatrix} = 21.891.$$

Here, the initial performance basis function influences the fitted value less than it did at trial 1; however, the asymptote and rate of change basis functions play more critical roles in deriving the fitted value at trial 5. The asymptote basis function is increasing in magnitude across the trials and the rate of change basis function is now decreasing in value although it is close to the point where it has maximal impact. Other fitted values across the trials could be computed in a similar way. It is most instructive to keep the overall shape of the basis functions in mind as a description of an individual's fitted function is assessed.

In contrast to individual 44 whose fitted function is neither monotonic nor exponential, individuals 55 and 74 show different patterns in their individual fitted functions. Unlike individual 44, the predicted person weights for individual 74 are $\hat{\boldsymbol{\eta}}_{74} = (17.928, 40.155, 0.432)'$. The person-level weights for the initial and asymptotic behavior have similar direction and magnitude as individual 44. In contrast, the weight for the rate of change for individual 74 is positive, whereas this weight for individual 44 is negative. This explains why the fitted trajectory for individual 74 behaves more like an exponential function relative to the fitted trajectory for individual 44 that does not. To see this more clearly, we computed the fitted function value at the initial trial for individual 74:

$$\hat{y}_{74,1} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 17.928 \\ 40.155 \\ 0.432 \end{pmatrix} = 17.928$$

The fitted values for individual 74 at subsequent trials are then computed as the additive effect of the linear combination with more weight given to the asymptote and rate basis functions in these trials relative to the first trial. In contrast, the fitted values for individual 44 following the initial trial are also dominated by the asymptote and rate basis functions but the rate weight is subtracted – not added. For individual 74, the fitted values at trials 5 through 9 are:

$$\hat{y}_{74,(5,\ldots,9)} = \begin{pmatrix} 43.901 \\ 44.823 \\ 44.807 \\ 44.503 \\ 44.431 \end{pmatrix}$$

Thus, the impact of the negative rate weight for individual 74 produces a fitted curve that looks more like an exponential function.

For individual 55, the person-weights produce another interesting case. For this individual, the fitted curve exhibits a nonconstant rate of change during the earlier trials, peaks at trial 3, and subsequently declines throughout the remainder of the trials. The person-level weights for individual 55 provides insight into this particular pattern. The predicted person-weights for individual 55 are $\hat{\boldsymbol{\eta}}_{55} = (26.294, 13.930, 1.103)'$. At the first trial, the initial person-weight dominates the fitted function, as it does for all individuals. As the influence of the initial basis function reduces at subsequent trials, the asymptote and rate basis functions become more dominant. For individual 55, however, relatively more weight is applied to the rate basis function. For instance, the rate weight is 0.432 for person 74, whereas the rate weight is 1.103 for person 55. In fact, the fitted function for individual 55 follows the same shape as the basis function for the rate parameter. We can see this by computing the fitted values for individual 55 using the values in Table 3 and the person-weights for the individual:

$$\hat{\mathbf{y}}_{55} = \begin{pmatrix} 1.00 & 0.00 & 0.00 \\ 0.74 & 0.27 & 14.99 \\ 0.54 & 0.46 & 21.97 \\ 0.40 & 0.60 & 24.15 \\ 0.29 & 0.71 & 23.59 \\ 0.21 & 0.79 & 21.61 \\ 0.16 & 0.84 & 19.00 \\ 0.12 & 0.88 & 16.24 \\ 0.09 & 0.92 & 13.60 \end{pmatrix} \begin{pmatrix} 26.294 \\ 13.930 \\ 1.103 \end{pmatrix} = \begin{pmatrix} 26.29 \\ 39.75 \\ 44.83 \\ 45.51 \\ 43.53 \\ 40.36 \\ 36.86 \\ 33.32 \\ 30.18 \end{pmatrix}$$

As shown in Figure 4, this similarity in shape is evident by plotting the fitted curve for individual 55 and the basis function for the rate parameter.

As a final comment, the idea that individuals’ fitted functions do not have to follow the target function is in stark contrast to fitting, for example, a nonlinear mixed-effects (NLME) model. For a NLME model, individuals’ fitted functions must adhere to the functional form used to describe the repeated measures but most likely with different values of the function parameters. That is, under a NLME model, all individuals are assumed to follow the same functional form but can vary with regard to any of the coefficients that describe change. As we showed here for a SLC model, there is no such restriction for the curves of the individuals. Thus, the fitted curves of the individuals may depart considerably from the form specified by the target function. This disparity between the two methods can be visualized in Figures 5 and 6. A small sample of individuals whose repeated measures data are well-described by an exponential function as in (10) are displayed in Figure 4. As shown, the fitted functions from fitting a SLC model and a NLME model fit the data for these individuals equally well. On the other hand, if the data for an individual follows a non-exponential trajectory, the fitted function computed from fitting a NLME model fits poorly; whereas a fitted function from a SLC model for the same individual fits remarkably well (see Figure 5). This comparison helps to highlight that the impact of the basis functions of a latent curve model on an individual's response can change over time. Understanding this difference between the two methods may be useful in deciding which of the two methods is preferred in practice. For the learning data presented here, the performance scores of the individuals were not expected to necessarily follow an exponential function, although the curves of many individuals did exhibit this pattern. Thus, the SLC model provided flexibility in accommodating the responses of those individuals whose performance did not follow the same form as the average response. Growth studies, in contrast, would most likely require that the individual curves follow a monotonic function, and so a NLME model for those types of data may be preferred.

Discussion

The process of formulating a model for a longitudinal response often involves selecting a function that includes one or more key features that describe the response in a useful way. For responses on a learning task, for example, Browne (1993) provides the parameterization of three functions with coefficients that have similar interpretations. More generally, a function may include a measure of time in addition to time-varying covariates (Davidian & Giltinan, 1995). In a latent curve model, formulation of a model begins by specifying a function at the individual level with the assumption that all individuals have this function in common. Across individuals in a population, the individual curves may vary in terms of the coefficients that are used to describe change. The structured latent curve model depends initially on the formulation of a model at the population level where it is the mean response that is assumed to follow a particular function. The model for the individual response depends on a first-order Taylor expansion of the target function that describes the mean response. In both the latent curve model and the structured latent curve model, the individual-level responses are assumed to be weighted linear combinations of a set of basis functions that are common to all members of the population.

This paper proposes the use of differential calculus to define and aid in the interpretation of the basis functions of a latent curve model. Similar to a structured latent curve model, the basis functions of a latent curve model can be defined using differential calculus. That is, the basis functions are defined by the first-order partial derivatives of the common function with regard to the change characteristics. Through this process, a separate function is defined for each slope associated with the change feature evaluated across time, holding constant other coefficients. Each basis function is defined as the change in the response with respect to a particular change feature. Thus, in addition to formulating a model with a particular function in mind, the individual basis functions may also be quite meaningful and relevant to the understanding of behavior. As was shown in the empirical example provided in this paper, it was useful to calculate the values of the basis functions, common to all individuals, at each measurement occasion to see how each basis function influences the fitted trajectories of the individuals. Additionally, calculation of the predicted person-weights provided a deeper understanding of how weights of different sign and magnitude could produce curves that differed in shape. Extending these ideas further, the basis functions themselves may be studied by examination of their partial derivatives with respect to time to obtain a measure of the sensitivity of the basis functions to time.

The use of nonlinear functions to characterize longitudinal responses under a latent curve model is becoming more common in the behavioral sciences. There is a restriction, however, in how structured latent curve models are specified to be considered a latent curve model in a strict sense. That is, the mean response function is assumed to be invariant to a constant scaling factor which consequently allows for a direct decomposition of the response function into a weighted combination of its basis functions. From this it is important to note that the first-order Taylor expansion yields a result that is not to be mistaken as a linear approximation to the specified function. Under a latent curve model this condition implies that the definition of a basis function is contingent on whether the corresponding model parameter enters the model in a linear manner. This is not a restriction of the structured latent curve model. Unlike a structured latent curve model, however, the expected values of the weights of a latent curve model are not restricted to certain values.

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Appendix

The target functions (e.g., exponential, logistic, Gompertz) that Browne (1993) introduced in his development of the SLCM were inherently nonlinear (at least one parameter entered the function in a nonlinear fashion) and monotonic. Each of the three functions, although having vastly different forms, were reparameterized to have coefficients that had the same interpretations. Specification of a SLCM, however, need not require that the target function adhere to strict monotoncity. For example, consider the following non-monotonic nonlinear function

$$f(t,\boldsymbol{\theta}) = (\beta_1 - \beta_2) \cdot t \cdot \exp(-\beta_3 t), \quad \text{where } \boldsymbol{\theta} = (\beta_1, \beta_2, \beta_3)'.$$

The derivatives of f with respect to elements of θ (i.e., the basis function of f) are

$$g_1 = f_1(t,\boldsymbol{\theta}) = \frac{\partial}{\partial\beta_1} f(t,\beta_1,\beta_2,\beta_3) = t \cdot \exp(-\beta_3 t)$$

$$g_2 = f_2(t,\boldsymbol{\theta}) = \frac{\partial}{\partial\beta_2} f(t,\beta_1,\beta_2,\beta_3) = -t \cdot \exp(-\beta_3 t)$$

$$g_3 = f_3(t,\boldsymbol{\theta}) = \frac{\partial}{\partial\beta_3} f(t,\beta_1,\beta_2,\beta_3) = -(\beta_1 - \beta_2) \cdot t^2 \cdot \exp(-\beta_3 t).$$

To demonstrate that f is invariant to a scaling factor is to show that there exists some positive scalar κ such that

$$f(\boldsymbol{\theta}^*) = \kappa f(\boldsymbol{\theta}).$$

That is, there ought to be a vector ζ such that

$$f(t,\boldsymbol{\theta}) = \boldsymbol{\Lambda\zeta} \quad \text{with} \quad \boldsymbol{\Lambda} = \frac{\partial}{\partial\boldsymbol{\theta}'} f(t,\boldsymbol{\theta}).$$

Using the invariance to a scaling factor property, we see that

$$\begin{aligned} f(\boldsymbol{\theta}^*) &= \kappa f(\boldsymbol{\theta}) \\ &= (\kappa\beta_1 - \kappa\beta_2) \cdot t \cdot \exp(-\beta_3 t), \quad \text{where } \boldsymbol{\theta}^* = (\kappa\beta_1, \kappa\beta_2, \beta_3)'. \end{aligned}$$

With the additional assumption that $\theta^* = \theta^*(\kappa,\theta)$ is differentiable with respect to κ (Satorra & Bentler, 1986), then taking the partial derivatives of both sides of $f(\theta^*) = \kappa f(\theta)$ with respect to κ evaluated at $\kappa = 1,$ it can be shown how to devise the elements of vector ζ so that $f(t,\theta) = \Lambda\zeta.$ That is,

$$\boldsymbol{\zeta} = \boldsymbol{\zeta}(\boldsymbol{\theta}) = \frac{\partial\boldsymbol{\theta}^*(\kappa,\boldsymbol{\theta})}{\partial\kappa}\bigg|_{\kappa=1} \begin{pmatrix} \kappa\beta_1 \\ \kappa\beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ 0 \end{pmatrix}.$$

Thus, the target function can be written as a latent curve model of the form $f(t,\theta) = \Lambda\zeta,$ given that $\zeta = \begin{pmatrix} \beta_1 & \beta_2 & 0 \end{pmatrix}'.$ For the non-monotonic target function above and a general row of Λ,

$$
\begin{aligned}
f(t,\boldsymbol{\theta}) &= \boldsymbol{\Lambda\zeta} \\
&= \begin{bmatrix} t \cdot \exp(-\beta_3 t) & -t \cdot \exp(-\beta_3 t) & -(\beta_1 - \beta_2) \cdot t^2 \cdot \exp(-\beta_3 t) \end{bmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ 0 \end{pmatrix} \\
&= (\beta_1 - \beta_2) \cdot t \cdot \exp(-\beta_3 t).
\end{aligned}
$$